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Plate-injection into a separated supersonic boundary layer

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The structure of a supersonic laminar boundary layer near a flat plate is examined when fluid is injected into it with velocity of $O(\epsilon^3 U_\infty^*)$ over a distance of $O(L)$. Here U_∞^* is the undisturbed fluid velocity, L the length of the plate and ϵ^{-8} is a representative Reynolds number. An essential requirement of the theory is that separation must have occurred upstream of the blow through a free interaction. It is assumed that between separation and the blow the reversed flow region has a wedge-like shape, of semi-angle in which $O(\epsilon^2)$, the fluid velocity has decayed to insignificant values at points just upstream of the blowing region. The blown fluid fills this wedge and the favourable pressure gradient necessary to drive this fluid downstream causes the boundary of the wedge to curve until at the end of the blow it is parallel to the plate. Explicit expressions for the pressure variation and boundary-layer thickness are worked out using a (crucially) modified form of the Cole–Aroesty theory. The relation between the strong injection studied here and massive injection, when the blowing velocity is of $O(U_\infty^*)$, is also discussed.

1. Introduction

This paper is concerned with the response of a laminar boundary layer, between a supersonic mainstream and a fixed wall, to a strong injection of fluid over an extended region of the wall. Suppose, to fix matters, that the wall is a finite flat plate maintained at a constant temperature and occupying the part $0 < x^* < L$ of the x^* axis of a Cartesian co-ordinate system Ox^*y^* and that Re, a representative Reynolds number of the flow, is large. Further, we take the mainstream velocity to be uniform except in so far as it is modified by the boundary layer: an external pressure gradient need not unduly complicate the flow field but here we wish to study the effect of injection in isolation.

Fluid is injected into the boundary layer from the plate with velocity V_w^* over the region $0 < x_0^* \leqslant x^* \leqslant x_1^* \leqslant L$. Superficially the simplest case is *weak* blowing and occurs when $V_w^* = O(Re^{-\frac{1}{2}} U_\infty^*)$, where U_∞^* is the undisturbed velocity of the fluid at an infinite distance upstream of the plate. The classical boundary-layer equations apply, provided separation does not occur, and the flow properties may be computed by direct integration. However, separation can occur if V_w^* or $x_1^* - x_0^*$ is large enough. For example if $x_0^* = 0$ the separation point is given by

$$x_s^* = O[(U_\infty^*/V_w^*)^2 L\, Re^{-1}] \tag{1.1}$$

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(Catherall, Stewartson & Williams 1965) and we can expect that if $x_0^* > 0$ the same V_w^* will produce separation at a smaller value of $x_s^* - x^*(> 0)$. Close inspection of the structural properties of the boundary layer calculated by Catherall *et al.* reveals that near separation it has the appearance of being blown off the plate and its subsequent behaviour is not yet known.

Strong blowing is usually defined rather loosely as occurring when

$$Re^{-\frac{1}{2}} \ll V_w^*/U_\infty^* \ll 1,$$

but in this paper we shall define it by the condition $V_w^* = O(Re^{-\frac{3}{8}}U_\infty^*)$ for we believe that this order of blowing velocity forms a key link and possibly the only one between the weak injection discussed above and massive injection where

$$V_w^*/U_\infty^* \sim 1.$$

A review of previous work on strong blowing has been given by Smith & Stewartson (1973) and so we shall content ourselves with observing that for the most part uniform injection velocities are excluded because the theories proposed then imply a pressure singularity at $x^* = x_0^*$. No mechanism has hitherto been suggested for producing the required pressure rise in $x^* < x_0^*$, nor for dealing with any separation which might result. Smith & Stewartson went on to resolve some of the difficulties of the earlier theories when the injection is through a narrow slot, specifically when $x_1^* - x_0^* = O(Re^{-\frac{3}{8}}L)$. The triple-deck theory of free interactions, developed by Stewartson & Williams (1969), plays a key role in their theory and leads to the conclusion that a self-induced pressure rise must occur upstream of the injection, when $\tilde{X} = (x^* - x_0^*)/\epsilon^3 L$ is negative and of $O(1)$, where

$$\epsilon = Re^{-\frac{1}{8}}. \tag{1.2}$$

Blowing starts at $\tilde{X} = 0$ and, in the calculations that were completed, the pressure p^* immediately begins to fall, continuing to do so until the termination of the blow at $\tilde{X} = \tilde{X}_1$, $\tilde{X}_1 = (x_1^* - x_0^*)/\epsilon^3 L^*$, after which it rises again to its original value p_∞^*, at least to within $O(\epsilon^6 p_\infty^*)$. The pressure at $\tilde{X} = 0$ can only be determined after a complicated numerical program because of the subtle interplay between p^*, V_w^* and $\tilde{X}_1$. No complete solutions were obtained for large values of $\tilde{X}_1 (\gg 15)$, but the available evidence suggests that adverse pressure gradients can then occur just after the start of blowing and might even be accompanied by separation with reattachment taking place further downstream. Smith & Stewartson were led to conjecture that for any $p^*(x_0^*) > p_\infty^*$, $V_w^* \to 0$ as $\tilde{X}_1 \to \infty$. Now from our earlier remarks on weak plate-blowing we can expect that $0 < x_s^* - x_0^* \leqslant O(U_\infty^{*2}L/Re\,V_w^{*2})$ and for strong blowing, in which

$$V_w^* = O(\epsilon^3 U_\infty^*), \quad 0 < \tilde{X}_s \leqslant O(\epsilon^{-1}),$$

where $\tilde{X}_s = (x_s^* - x_0^*)/\epsilon^3 L$. It may well be in fact that then $\tilde{X}_s = O(1)$. Thus a supersonic boundary layer separates after a blowing distance of $O(L)$ if the injection velocity is weak, so that no interaction occurs between the boundary layer and the mainstream. As the blowing rate is increased, separation moves upstream until, with strong injection, it occurs within a distance of $O(\epsilon^3 L)$ of the onset of blowing.

The line of thought is superficially attractive, and indeed the incomplete investigations of slot blowing confirm that separation would occur when $\tilde{X} = O(1)$

in strong plate blowing, but we are not as yet inclined to accept it unreservedly. Our caution arises from the difference in the mode of appearance of the separation in the two situations. The implication of the conventional plate-injection theory is that $\tilde{X}_s$ is a decreasing function of V_w^*, being large when $V_w^* \epsilon^{-3} U_\infty^{*-1}$ is small, whereas in the slot-injection studies separation is only found to occur near $\tilde{X} = 0$. Perhaps it is possible to reconcile these conclusions but, on the other hand, it may be that in supersonic boundary layers all separations are characterized by free interactions. In that event classical boundary-layer theory only holds if V_w^* is too small to provoke separation. Once V_w^* is increased to a point where separation occurs (at x_1^*) a discontinuity in the evolution of the flow field occurs and the separation point moves to a point very near x_0^*, i.e. near the start of the blowing. There is certainly some obscurity about the transition from weak plate-injection to strong slot-injection which needs further study for its clarification.

Of more importance for the present paper is the other limitation of the slot-blowing study, that the skin friction at $\tilde{X} = 0$ should be positive, i.e. that separation should not have occurred before the onset of blowing. It is clear from the numerical investigation that this requirement defines a curve in $\tilde{X}_1$, $V_w^*/\epsilon^3 U_\infty^*$ space beyond which a new procedure is needed to elucidate the boundary-layer structure. The reason is that a region of reversed flow (see figure 1) is set up between the separation point $\tilde{X}_s$ and $\tilde{X} = 0$, for which the techniques used hitherto are strictly inadequate. The singularity at $\tilde{X} = 0$, which required so much care, will also need modification and may even disappear; it is also conceivable but unlikely that the reversed flow extends far beyond $\tilde{X} = 0$. Separation itself does not present an insuperable mathematical problem – the singularity which is so much a part of classical theory cannot occur here.

We shall not attempt to resolve the numerical questions here but instead consider the flow properties when the blowing is such that $-\tilde{X}_s$ is large and positive. We can then make use of the important numerical study of the separated part of the free-interaction region due to Mr P. G. Williams, and which he has kindly made available to us. This study indicates that the further continuation of the free-interaction solution, presented by Stewartson & Williams (1969), into the reversed flow region, gradually develops an asymptotic structure. In it the pressure is constant, the lateral extent of the reversed flow region is increasing linearly with $\tilde{X}$ and the velocity in this part of the flow field is increasing towards zero. This structure is broadly in agreement with the experimental studies of free-interaction flow (i.e. Chapman, Kuehn & Larson 1958), although the observed value of the plateau pressure is somewhat smaller than that predicted by Williams (see Stewartson & Williams 1969). The likely reason is that the theory has a relative error of $O(\epsilon)$, whereas the experiments were carried out at values of $\epsilon \approx 0{\cdot}25$.

In this paper we shall suppose that there is enough blowing for the pressure to have reached its plateau value at $x^* = x_0^*$ and the thickness δ_0^* of the reversed flow region satisfies $\delta_0^* \gg \epsilon^4 L$, i.e. is much thicker than the undisturbed boundary layer, while remaining small compared with L. The condition on x_s^* is that $\epsilon^2 L \ll x_0^* - x_s^* < L$, which is acceptable on physical grounds. The injected fluid immediately spreads right across the reversed flow region without a significant part of the main flux penetrating upstream of x_0^*. The outer boundary then bends back parallel to the plate and so produces a favourable pressure gradient which drives the blown fluid downstream. The equation determining the pressure is closely similar to that found by Cole & Aroesty (1968), but the modifications introduced by the use of Williams asymptotic solution enable us to overcome the difficulties in their theory when applied to uniform injection (see Smith 1972). The theory is not complete, however, for the details of the transition from the reversed flow to the blown flow, presumably occurring in a distance $\leqslant O(\epsilon L)$, are left unresolved. In addition, the pressure fluctuation occurring at the end of the blowing region (near $x^* = x_1^*$) is not discussed in detail. Our expectation is that the resolution of these aspects will not disturb the main conclusions of this paper.

It is interesting to note, however, that the incomplete calculations of slot blowing with X_1 large, reported by Smith (1972) and Smith & Stewartson (1973), in which small regions of reversed flow were found, contain in embryo the structure of the flows assumed in the present plate-injection studies.

2. The free-interaction region and the Williams asymptotic profile

We shall suppose that the fluid is Newtonian with coefficients of viscosity μ and thermal conductivity k functions of temperature T only. The pressure at any point is denoted by p^*, the density by ρ^* and the components of velocity parallel to the (x^*, y^*) axes by (u^*, v^*) respectively; * signifies a dimensional quantity and we denote by the suffixes ∞ and w conditions at infinity and on the plate. An implied and significant condition is that the plate temperature should be the same everywhere. If there is a substantial change in temperature, at x_0^* for example, the theory needs reconsideration. The free-interaction region is supposed to be centred on the point $(x_s^*, 0)$ where the skin friction vanishes and we define

$$Re = U_\infty^* x_s^* / \nu_\infty = \epsilon^{-8}, \tag{2.1}$$

ν being the kinematic viscosity.

The local structure of the free-interaction region when $\epsilon \ll 1$ is explained in Stewartson & Williams (1969), to which the reader is referred for a detailed discussion. Briefly, it is confined within a streamwise distance of $O(\epsilon^3 x_s^*)$ of $(x_s^*, 0)$ and has a three-decked structure of which the lowest is of thickness of $O(\epsilon^5 x_s^*)$, and the flow properties therein satisfy the incompressible boundary-layer equations

$$U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Z} = -\frac{dP}{dX} + \frac{\partial^2 U}{\partial Z^2}, \quad \frac{\partial U}{\partial X} + \frac{\partial V}{\partial Z} = 0, \tag{2.2}$$

where
$$x^* = x_s^* + \epsilon^3 x_s^* \frac{C^{\frac{3}{8}}(T_w/T_\infty)^{\frac{3}{2}}}{\lambda^{\frac{5}{4}}(M_\infty^2 - 1)^{\frac{3}{8}}} X, \quad y^* = \epsilon^5 x_s^* \frac{C^{\frac{5}{8}}(T_w/T_\infty)^{\frac{3}{2}}}{\lambda^{\frac{3}{4}}(M_\infty^2 - 1)^{\frac{1}{8}}} Z,$$

$$u^* = \epsilon U_\infty^* C^{\frac{1}{8}}(T_w/T_\infty)^{\frac{1}{2}} \lambda^{\frac{1}{4}}(M_\infty^2 - 1)^{-\frac{1}{8}} U(X, Z),$$

$$v^* = \epsilon^3 U_\infty^* C^{\frac{3}{8}}(T_w/T_\infty)^{\frac{1}{2}} \lambda^{\frac{3}{4}}(M_\infty^2 - 1)^{\frac{1}{8}} V(X, Z)$$

and
$$p^* = p_\infty^* + \epsilon^2 \rho_\infty^* U_\infty^{*2} C^{\frac{1}{4}} \lambda^{\frac{1}{2}}(M_\infty^2 - 1)^{-\frac{1}{4}} P(X). \tag{2.3}$$

M_∞ is the Mach number of the flow, $\lambda = 0{\cdot}332\ldots$ is the skin-friction factor inferred from the Blasius equation and $C = \mu_w T_\infty / \mu_\infty T_w$ is the Chapman function. This quantity C admittedly only represents approximately the effect of a general viscosity–temperature law but is generally believed to be adequate in a wide variety of circumstances.

In addition to (2.2) the following boundary conditions must be satisfied:

$$U = V = 0 \quad \text{at} \quad Z = 0 \tag{2.4a}$$

(the no-slip condition);

$$U - Z \to 0 \quad \text{as} \quad X \to -\infty \tag{2.4b}$$

(to match up with the Blasius profiles upstream of the interaction);

$$U - Z \to A(X) \quad \text{as} \quad Z \to \infty, \tag{2.4c}$$

where

$$A'(X) = -P(X). \tag{2.4d}$$

This final condition arises because the increase in displacement thickness of the boundary layer as separation is approached induces a pressure gradient in the mainstream which in turn provides the mechanism for driving the inner boundary layer. Between the lower deck of the boundary layer and the inviscid mainstream, the main deck of the boundary layer responds passively and the changes in it can, to leading order, be regarded simply as a translation outwards of its streamlines.

Clearly there is a trivial solution of (2.2) satisfying (2.4), namely $U = Z$, but it is not unique. In Stewartson & Williams (1969) another solution was found in which $U - Z$ and P are exponentially small when X is large and negative. As X increases P increases and the skin friction

$$\tau^* = \left(\mu \frac{\partial u^*}{\partial y^*}\right)_w = \frac{\lambda C^{\frac{1}{2}}}{x_s^*} \mu_\infty U_\infty^* \left(\frac{\partial U}{\partial Z}\right)_w \tag{2.5}$$

falls until eventually separation is reached and $\tau \equiv (\partial U/\partial Z)_w = 0$. This point is the chosen origin of X and strictly speaking the integration can be carried no further because the existence of negative values of U on the lines $X = \text{constant} > 0$ implies that the boundary-value problem in (2.2) and (2.4) is not well posed. Upstream of $X = 0$ the solution of (2.2) found appears to be unique (granted that there is a separation point), stable and relatively easy to compute.

In $X > 0$, Mr Williams has conducted a number of numerical experiments in order to elucidate the properties of (2.2). The use of a step-by-step integration procedure in the direction of X increasing springs most readily to mind but the absence of a downstream boundary condition when $U < 0$ implies a non-uniqueness in the mathematical solution and a tendency for instability in the numerical work. An alternative is the solution of a time-dependent set of equations in the hope that in the limit of infinite time the steady state emerges. Here again, however, the need soon becomes apparent for a terminal condition on U, when X takes on some positive value X_2, at those values of Z at which U turns out to be negative. A similar situation arises in a related study by Belcher, Burggraf & Stewartson (1972). A third possibility is to use the Flügge-Lotz & Reyhner (1968) approximation in which $U\,\partial U/\partial X$ is neglected when $U < 0$.

X	5	10	15	20	25	30	35	40
P	1·62	1·72	1·72	1·78	1·79	1·80	1·80	1·80
τ	−0·127	−0·078	−0·052	−0·040	−0·032	−0·026	−0·022	−0·019
$U_{\min}$	−0·201	−0·261	−0·273	−0·277	−0·272	−0·266	−0·259	−0·253

TABLE 1

Mr Williams's numerical studies, while not complete, strongly suggest that (2.2) possesses a solution in which

$$P \to P_0 \quad \text{as} \quad X \to \infty, \tag{2.6a}$$

where P_0 is a numerical constant approximately equal to 1·8. Again if Z/X tends to a limit χ as $X \to \infty$

$$U \to 0- \quad \text{if} \quad \chi < P_0, \tag{2.6b}$$

$$\partial U/\partial Z \to 1 \quad \text{if} \quad \chi > P_0. \tag{2.6c}$$

In the neighbourhood of $\chi = P_0$ there is a shear layer of thickness $\propto X^{\frac{1}{3}}$ whose properties seem to be approaching those of the similarity solution

$$\left.\begin{aligned} &U = X^{\frac{1}{3}}F'[(Z-P_0X)/X^{\frac{1}{3}}], \quad F^{\mathrm{iv}}+\tfrac{2}{3}FF''' = 0,\\ &F''(\infty) = 1, \quad F'(-\infty) = 0. \end{aligned}\right\} \tag{2.7}$$

In table 1 we display representative calculations of P, τ and $U_{\min}$ at various values of X, when $U_{\min}$ is the minimum value of U regarded as a function of Z. They were computed using the approximate method of Flügge-Lotz & Reyhner. The support for (2.6) is quite firm from P and τ, both of which are clearly approaching their required limits, and in particular $\partial P/\partial X$ appears to be rapidly tending to zero with increasing X. On the other hand, the variation of $U_{\min}$ is slow when X is large and there must be some doubt as to whether $U_{\min} \to 0$ as $X \to \infty$. The mass flux across the reversed flow region

$$\delta^*_- = -\int_0^{Z_1} U\,dZ, \quad \text{where} \quad U(X, Z_1) = 0, \tag{2.8}$$

is still increasing at the termination of the forward integration with respect to X. This is not unexpected for, in order to match with the shear layer (2.7), we must have $\delta^*_- \propto X^{\frac{2}{3}}$ since δ^*_- is the (minimum) value of the stream function at $Z = Z_1$ but it is not clear from the numerical studies whether $\delta^*_-/X^{\frac{2}{3}}$ is approaching a limit. If it were then presumably $U_{\min} \propto X^{-\frac{1}{3}}$ from (2.8) which would account for its slow rate of decay in table 1.

The uncertainty about the limiting structure would be removed if a full asymptotic description of the solution could be obtained, but this has not proved possible.† Nevertheless, it is difficult to see what the alternative might be in view of the above discussion, and (2.6) seems to be in line with observations of the deadwater region in the interactions between shock waves and boundary layers. Also we know from related studies of the Falkner–Skan equation (Stewartson 1954; Cebeci & Keller 1971) that a very weak adverse pressure gradient can induce a surprisingly large reversed velocity even though there is none in the limit as the adverse pressure gradient approaches zero.

† See *Note added in proof*.

Strictly speaking any reversed velocity profile is possible at the terminal station X_∞ because of the nature of the governing parabolic equation. In general the corresponding solution will be far from similar anywhere in the range $0 < X \leqslant X_\infty$. In order to be able to match this solution to that in the injection region we need a solution that is essentially self-preserving. It seems that (2.6) has this property. The mainstream velocity is uniform and the boundary layer develops as a mixing region between that flow and the virtually stagnant flow beneath. Further, its rate of spreading ($\propto X^{-\frac{2}{3}}$) is slower than that of the uniform rate at which it is separating from the plate. We anticipate therefore that its principal features would be retained when X is so large that $x^* - x_s^* \sim L$.

From the point of view of the blown fluid the uniform pressure in the reversed flow region when $X \gg 1$ and the dragging effect of the mainstream flowing above it make it unlikely that a significant fraction will initially move upstream. We are confident that the assumption of the Williams asymptotic profile (2.6) as an initial profile for the blown fluid is correct at best and a good approximation at worst.

3. The region of blowing

Fluid is blown into the boundary layer from the plate with uniform velocity V_w^* and density ρ_w^*, starting at $x^* = x_0^* > 0$ and stopping at $x^* = x_1^* < L$. These imposed conditions are assumed to be sufficient to ensure that the boundary layer has separated at $x^* = x_s^* < x_0^*$ and that at $x^* = x_0^*$ the separated flow is fully established and the Williams asymptotic profile holds. Hence at $x^* = x_0^*$

$$\left.\begin{aligned} u^* &= \begin{cases} 0 & \text{for} \quad 0 < y^* < \delta_0^*, \\ U_\infty^* - \epsilon^2 U_\infty^* \lambda^{\frac{1}{2}} C^{\frac{1}{4}} (M_\infty^2 - 1)^{-\frac{1}{4}} P_0 & \text{for} \quad y^* > \delta_0^* + \bar{\delta}^*, \end{cases} \\ p^* &= p_\infty^* + \epsilon^2 \rho_\infty^* U_\infty^{*2} \lambda^{\frac{1}{2}} C^{\frac{1}{4}} (M_\infty^2 - 1)^{-\frac{1}{4}} P_0. \end{aligned}\right\} \tag{3.1}$$

Here δ_0^* is an unknown physical constant to be found and is related to the distance upstream of the blowing zone at which separation occurs, but we shall require $\delta_0^*/L \gg \epsilon^4$. Asymptotically,

$$\delta_0^* = \epsilon^2 \lambda^{\frac{1}{2}} C^{\frac{1}{4}} (M_\infty^2 - 1)^{\frac{1}{4}} P_0 (x_0^* - x_s^*), \tag{3.2}$$

using (2.3) and (2.4d). Again, $\bar{\delta}^*$ is the thickness of the shear layer separating the inviscid mainstream and the stagnant fluid and so $\bar{\delta}^* \sim \epsilon^4 L$. For the success of the theory that we shall develop $\delta_0^* \gg \bar{\delta}^*$, implying

$$x_0^* > (x_0^* - x_s^*) \gg \epsilon^2 L, \tag{3.3}$$

so that in physical terms the distance between the blow and the separation point upstream can still be very small. Finally we note that there must be some blurring at the dividing lines $y^* = \delta_0^*$, $\delta_0^* + \bar{\delta}^*$ as is usual in boundary-layer theory but this is of no significance in the present context.

The blown fluid fills the whole region $0 < y^* < \delta_0^*$ at $x^* = x_0^*$ and behaves as if it were at a forward stagnation point, so that at $y^* = \delta_0^*$, $x^* = x_0^*$ it is at rest. Since fluid is entering the boundary layer for $x^* > x_0^*$ the fluid must move downstream and hence a favourable pressure gradient must be developed. The pressure

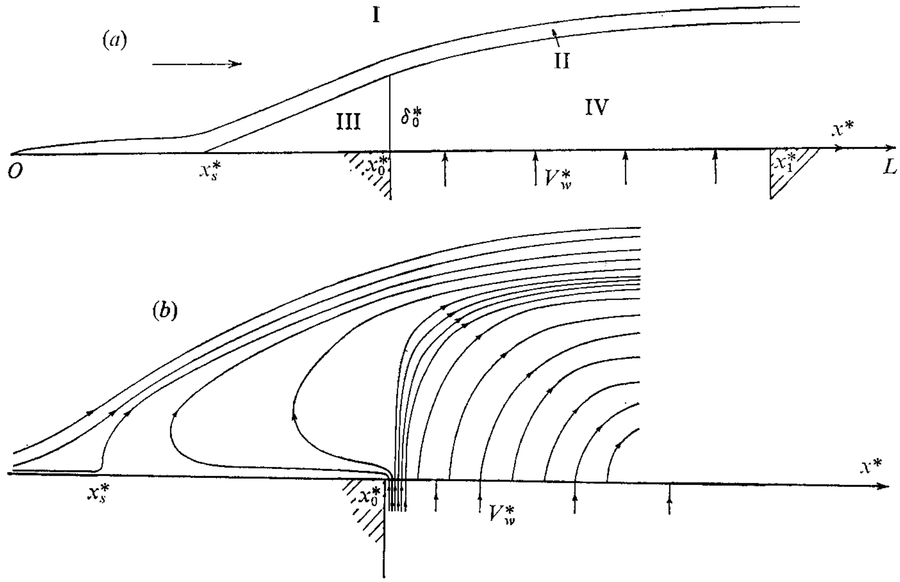

FIGURE 1. (a) Schematic diagram of the flow model for strong plate-injection. Region I is the inviscid mainstream, II the detached viscous boundary layer, III is a region of slow reversed flow between x_s^* and x_0^*, and IV is the main inviscid blown region $x_0^* \leqslant x^* \leqslant x_1^*$. Region IV is initially of height $\delta_0^* \gg L\,Re^{-\frac{1}{2}}$, where L is the plate length. (b) Description of the probable streamline pattern when the boundary layer separates ahead of the blow.

therefore decreases towards the value p_∞^*, which it must attain at large distances downstream of the plate, at least to within $O(\rho_w^* V_w^*/\rho_\infty^* U_\infty^*)$ (see below). The main regions of the flow field envisaged are sketched in figure 1 (a), and in figure 1 (b) a plausible description of the streamlines, particularly near $x^* = x_s^*$, is sketched. It is assumed in figure 1 (b) that the initial part of the injected fluid moves upstream and then is entrained into the free shear layer, which is the continuation of the original boundary layer beyond separation, and which overlies the injected fluid. Whether we make use of normal entrainment principles or of the amount implied by (2.7) $[F(-\infty)]$ it appears that a mass flux of $O(\rho_\infty^* \epsilon^4 U_\infty^* L)$ is needed when $x^* - x_s^* \sim L$. Thus a length of injection of $O(\epsilon L)$ fulfils the requirement and is negligible in comparison with the total length $x_1^* - x_0^*$ of blowing.

Define the thickness of the boundary layer including the blown fluid (II + IV) to be $\delta^*(x^*)$ so that

$$\delta^*(x^*) = \delta_0^* \quad \text{at} \quad x^* = x_0^*, \tag{3.4}$$

where $\bar{\delta}^*$ is neglected in comparison with δ_0^*. The pressure in the mainstream is then given by

$$p^* = p_\infty^* + (M_\infty^2 - 1)^{-\frac{1}{2}} \rho_\infty^* U_\infty^{*2} (d\delta^*/dx^*). \tag{3.5}$$

Finally the boundary layer of the blown fluid (region IV) may be assumed to be incompressible and inviscid (Cole & Aroesty 1968) provided that $V_w^*/\epsilon^4 U_\infty^* \gg 1$. The governing equations for the fluid in region IV are then, since $0 < \delta^*(x^*) \ll L$,

$$\rho_w^* \left(u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} \right) = -\frac{\partial p^*}{\partial x^*}, \quad \frac{\partial p^*}{\partial y^*} = 0, \quad \frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0. \tag{3.6}$$

Here, u^*, v^* and $p^*-p^*_\infty$ are of orders ϵU^*_∞, $\epsilon^3 U^*_\infty$ and $\epsilon^2\rho^*_\infty U^{*2}_\infty$ respectively, with $x^*-x^*_0$ and y^* being of $O(L)$ and $O(L\epsilon^2)$ respectively. The boundary conditions imposed in IV are:

$$\left.\begin{aligned} u^* = 0 \quad \text{at} \quad y^* = 0, x^* > x^*_0; \quad u^* = 0 \quad \text{at} \quad x^* = x^*_0, y^* > 0; \\ v^* = V^*_w \quad \text{at} \quad y^* = 0, x > x^*_0; \quad v^*/u^* = d\delta^*/dx^* \quad \text{at} \quad y^* = \delta^*(x^*);\end{aligned}\right\} \tag{3.7}$$

the last condition in (3.7) being necessary because region II has a thickness negligible in comparison with that of IV and so to first order acts as the dividing streamline between the injected and mainstream fluids. The solution of (3.6) subject to (3.7) is

$$\delta^*(x^*) = V^*_w\left(\frac{\rho^*_w}{2}\right)^{\frac{1}{2}}\int_{x^*_0}^{x^*}\frac{dt}{[p^*(t)-p^*(x^*)]^{\frac{1}{2}}}, \tag{3.8}$$

from Cole & Aroesty (1968), and follows on using the theory of strong blowing in a favourable pressure gradient given by Gadd, Jones & Watson (1963).

Thus the determination of the principal properties of the blown boundary layer is provided by the solution of (3.5) and (3.8) together with the initial conditions implied by (3.1) and (3.2). Apart from these conditions, the equations are identical to those proposed by Cole & Aroesty (1968), but the difference is crucial to the success of the theory when applied, as here, to uniform blowing. For unless $\delta^* \neq 0$ at the start of blowing the only way to achieve a solution of (3.8) and (3.6) with V^*_w constant is to assume that $p^*-p^*_\infty \propto (-\log[x^*-x^*_0])^{\frac{1}{2}}$ so that p^* has a logarithmic singularity at $x^* = x^*_0$, which implies that the boundary layer has already separated. If x^*_0 is taken to be zero to avoid this possibility then the pressure rise still remains unaccounted for and its elucidation through a study of the full Navier–Stokes equations near the leading edge is a formidable task. By accepting the separation of the boundary layer and using the Williams asymptotic formula we reduce the study of the blown boundary layer to the solution of a manageable integro-differential equation.

Let us introduce the transformations

$$p^* = p^*_\infty + \epsilon^2\rho^*_\infty U^{*2}_\infty \lambda^{\frac{1}{2}} C^{\frac{1}{4}}(M^2_\infty-1)^{-\frac{1}{4}} P_0 p(x),$$

$$\delta^*(x^*) = \delta^*_0\Delta(x), \quad x^* = (\delta^*_0/\epsilon^2)(M^2_\infty-1)^{-\frac{1}{4}}\lambda^{-\frac{1}{2}}C^{-\frac{1}{4}}x/P_0 + x^*_0 \tag{3.9a}$$

and write

$$\kappa = \frac{V^*_w}{\epsilon^3 U^*_\infty}\left(\frac{\rho^*_w}{2\rho^*_\infty P^3_0}\right)^{\frac{1}{2}}(M^2_\infty-1)^{-\frac{1}{8}}\lambda^{-\frac{3}{4}}C^{-\frac{3}{8}} \equiv \frac{V_w\pi^{\frac{1}{2}}}{(2P^3_0)^{\frac{1}{2}}}. \tag{3.9b}$$

In terms of these new variables the equation for $p(x)$ reduces to

$$\Delta(x) = \frac{\kappa}{\pi^{\frac{1}{2}}}\int_0^x \frac{dt}{[p(t)-p(x)]^{\frac{1}{2}}}, \quad \Delta'(x) = p(x), \tag{3.10}$$

with initial conditions

$$\Delta(0) = 1, \quad \Delta'(0) = 1 = p(0). \tag{3.11}$$

For plate-injection the length of blowing $(x^*_1-x^*_0)$ is $O(L)$ and hence $\delta^*_0 = O(\epsilon^2 L)$. It follows that the significant scaling in the blowing velocity is $V^*_w/U^*_\infty = O(\epsilon^3)$, for this permits the integro-differential equation to be cast in a form independent of ϵ, κ being then of $O(1)$. Separation now occurs at a distance of $O(L)$ upstream of x_0^*.

It is gratifying to find that not only do the orders of magnitude of u^*, v^* and p^* in the blown region IV coincide with those of the lower deck, equation (2.3), but the determination of the flow properties in IV reduces to a problem in which the only remaining parameter is V_w. The first finding is consistent with the envisaged flow pattern, in which the reversed flow region that starts in the lower deck joins smoothly (in terms of pressure and velocity) onto the forward flow region at $x = o(1)$ and is then filled with injected fluid, while the fundamental parameter V_w is exactly that found in the companion paper (Smith & Stewartson 1973). Thus, the blowing length apart, it is V_w alone that determines whether the flow description is as presented here or as in the companion paper (or perhaps belongs to an intermediate regime) and we can therefore expect a transition from the one to the other as V_w is decreased.

As stated the solution of (3.10) subject to (3.11) is unique as we shall see in the next section. However, it cannot be extended to $x = \infty$ because then a simple asymptotic analysis, again verified by the complete solution to be obtained below, shows that $p \propto -(\log x)^{\frac{1}{2}}$ and hence p cannot be directly joined to the uniform pressure which one might expect to be operative downstream of the plate.

In order to fix an appropriate condition on p let us take the plate in a wind tunnel whose width is comparable with L. Then owing to the injection of fluid the mass flux of fluid down the tunnel will be increased by a factor $1+O(\rho_w^* V_w^*/\rho_\infty^* U_\infty^*)$ i.e. by a factor $1+O(Re^{-\frac{3}{8}})$. Hence the pressure will be diminished by a factor $1-O(Re^{-\frac{3}{8}})$ and so using (3.9) $p = 0$ downstream of the plate. Accordingly we take

$$p(x_1) = 0,$$

where
$$x_1 = \frac{\lambda^{\frac{5}{4}} C^{\frac{3}{8}} (M_\infty^2 - 1)^{\frac{1}{4}} P_0}{\delta_0^* Re^{\frac{1}{4}}} (x_1^* - x_0^*), \tag{3.12}$$

and is the terminal point of the blowing. This condition assumes that the termination of the blowing modifies the solution in the neighbourhood of $x = x_1$ only and that the principal properties of the flow are continuous through this point. The condition (3.12) seems at first sight to be reasonable since the fluid is moving very slowly in the whole of the blown region, in contrast to the situation at the free interaction. The effect of terminating the blow when the blow is massive has been considered by Fernandez & Lees (1970), Taylor, Masson & Foster (1969) and Thomas (1969), all using integral methods. A cusp- or sink-like behaviour occurring in the solutions was attributed to the effect of stopping the blow, but we are inclined to be cautious in drawing conclusions from their results as it is not clear whether the singularities are indeed representative of the physics of the flow near $x = x_1$ or rather are a shortcoming of the integral method. Taylor *et al.* studied blowing from a cone, with the blow terminating at its base. Here the sharp corner causes an expansion fan and they make the reasonable assumption that the inviscid flow is then sonic. The justification for the pressure condition we have imposed ($p(x_1) = 0$) in the present problem requires a detailed study of the properties of the solution in the neighbourhood of $x = x_1$. We shall report on this work elsewhere, but note here that the pressure change at the termination of the blow is of $O(\epsilon^4 \rho_\infty^* U_\infty^{*2})$ and occurs in a characteristic length of $O(\epsilon^2 L)$ in contrast to the results of Fernandez & Lees (1970).

4. The determination of δ_0^*

The only unknown in the problem as posed by (3.9)–(3.11) is δ_0^*, effectively the length of the reversed flow bubble from (3.2), and this we may find by determining x_1 as a function of κ and using (3.12).

When x is small we may write

$$p(x) = 1 - \alpha_2 x^2 + \alpha_3 x^3 + \ldots, \tag{4.1}$$

whence on substituting into (3.10) and using (3.11) we find that

$$\alpha_2 = \tfrac{1}{4}\kappa^2 \pi, \quad \alpha_3 = \tfrac{1}{8}\kappa^2 \pi^2, \tag{4.2}$$

so that both the pressure and pressure gradient are continuous at $x = 0$. Further, the streamwise velocity in the blown region is proportional to x when $x \ll 1$, so that the point $x = 0$ behaves like a forward stagnation point and the velocity is continuous there. The axially symmetric analogy of the flow near $x = 0$ has been discussed, experimentally and theoretically, by Libby (1962). We also see that when κ is large

$$x_1 \approx 2/\kappa \pi^{\frac{1}{2}}, \tag{4.3}$$

so that

$$\delta_0^* \approx \frac{(x_1^* - x_0^*)(M_\infty^2 - 1)^{\frac{1}{4}}}{2Re^{\frac{1}{4}}\lambda^{\frac{1}{4}} C^{\frac{1}{8}}} \left(\frac{V_w^* Re^{\frac{3}{8}}}{U_\infty^*}\right)\left(\frac{\rho_w^*}{2\rho_\infty^* P_0}\right)^{\frac{1}{2}}, \tag{4.4}$$

whence

$$\frac{x_0^* - x_s^*}{x_1^* - x_0^*}\left(\frac{V_w^* Re^{\frac{3}{8}}}{U_\infty^*}\right)^{-1} = \text{constant}. \tag{4.5}$$

Thus when the blowing is very strong ($V_w \gg 1$, i.e. $V_w^* \gg \epsilon^3 U_\infty^*$) the length of the separated region is formally linearly proportional to the length of the blowing region and to the blowing velocity. However, x_s^* is limited by the condition of $x_s^* > 0$ and so as V_w increases (4.5) eventually leads to a contradiction. We note that $\epsilon = (\nu_\infty / x_s^* U_\infty^*)^{\frac{1}{8}}$ and so once x_s^* becomes small a further increase of V_w^* can be reflected in an increase of ϵ rather than a significant change in V_w or $x_0^* - x_s^*$. Again when $\epsilon \sim 1$ the boundary-layer assumptions fail and separation occurs in a region controlled by the full Navier–Stokes equations. Further, the downstream condition on the pressure (3.12) fails when the blowing is hard enough that the relative increase in mass flux is of the same order as the relative pressure rise on the plate. We cannot estimate this since, when $\epsilon \sim 1$, we do not know the dependence of the plateau pressure on ϵ, as the boundary-layer assumptions no longer hold.

It seems nevertheless that a link can be formed with the massive blowing problem in which $V_w^* \sim U_\infty^*$. From the above discussion we can expect that separation then occurs at the leading edge of the plate and a largely stagnant wedge of fluid is formed upstream of the blown region. Thus the flow configuration might well be similar to that in the well-known aerodynamic spike (see, for example, Birkhoff 1960, Frontispiece). The blown fluid then forms a finite body and the presence of part of the plate in front of this body ensures that the oncoming fluid separates at the leading edge, causing a wedge of stagnant fluid above the plate and upstream of the blown fluid. Further, since the rise in the pressure on the plateau is of $O(\epsilon^2 p_\infty^*)$ in strong blowing and ϵ increases without limit as $x_s^* \to 0$, it is possible in principle for this rise to be of $O(p_\infty^*)$ in massive blowing, in accord with the expectation that the resulting flow pattern is largely inviscid in character. Although this flow picture for massive blowing looks reasonable, granted the reality of the spike, the experimental evidence available to us is somewhat equivocal. Hartunian & Spencer (1967) investigated massive blowing at $M_\infty = 4{\cdot}5$ and a Reynolds number based on x_0^* of about 200. There were some signs of separation ahead of x_0^* on a wedge but no sign on a cone. On the other hand, Bott (1968) at the same Mach number but a lower value of Reynolds number (~ 70) found evidence of a shock emanating from $x^* = \frac{1}{3}x_0^*$ ($= \frac{1}{8}$ in.), which might therefore be interpreted as a separation point. In both cases the Reynolds number was far too small for the subtle nuances of weak, moderate or strong blowing to have much significance. There is also a large body of theoretical work on massive blowing, chiefly semi-empirical, to which the interested reader can refer; for example Inger & Gaitatzes (1971), who include a useful list of references.

The explicit evaluation of x_1 as a function of κ can be effected by changing the independent variable from x to p and writing

$$dx/dp = -G(p). \tag{4.6}$$

We then have

$$1 + \int_p^1 qG(q)\,dq = \frac{\kappa}{\pi^{\frac{1}{2}}}\int_p^1 \frac{G(q)\,dq}{(q-p)^{\frac{1}{2}}} \tag{4.7}$$

with

$$G(p)(1-p)^{\frac{1}{2}} \to 1/\kappa\pi^{\frac{1}{2}} \quad \text{as} \quad p \to 1.$$

Now write

$$G = \sum_{n=0}^{\infty} (1-p)^{\frac{1}{2}(n-1)} A_n, \quad A_0 = 1/\kappa\pi^{\frac{1}{2}}; \tag{4.8}$$

then on substituting into (4.7) we can easily form a difference equation for the A_n, namely

$$A_{n+1} = \frac{(\frac{1}{2}n-\frac{1}{2})!}{(\frac{1}{2}n)!\,\kappa}(A_n - A_{n-1}), \tag{4.9}$$

taking $A_{-1} = A_{-2} = 0$. Having determined all the A_n using (4.9), or to be precise a derived form obtained by applying it to A_n and A_{n-1} in (4.9) and so eliminating the factorials by forming a four-term difference equation, x_1 follows on integration of (4.8), i.e. from

$$x_1 = \sum_{n=0}^{\infty} \frac{2A_n}{n+1}. \tag{4.10}$$

Values of x_1 as a function of κ are given in table 2. The variation of p with x/x_1 is shown in figure 2 for various values of κ.

The asymptotic behaviour of x_1 when κ is small, and indeed a formal solution for G in terms of p, can be found by reducing the Abel equation in (4.7) to a

κ	5	4	3	2	1	0·9	0·8	—
x_1	0·259	0·336	0·476	0·813	2·532	3·142	4·088	
κ	0·7	0·6	0·5	0·4	0·3	0·2	0·1	0
x_1	5·701	8·87	16·68	45·80	316·0	48,980	—	∞
$x_1\kappa\, e^{-1/3\kappa^2}$	2·021	2·108	2·199	2·281	2·335	2·355	(2.363)	2·364

TABLE 2.

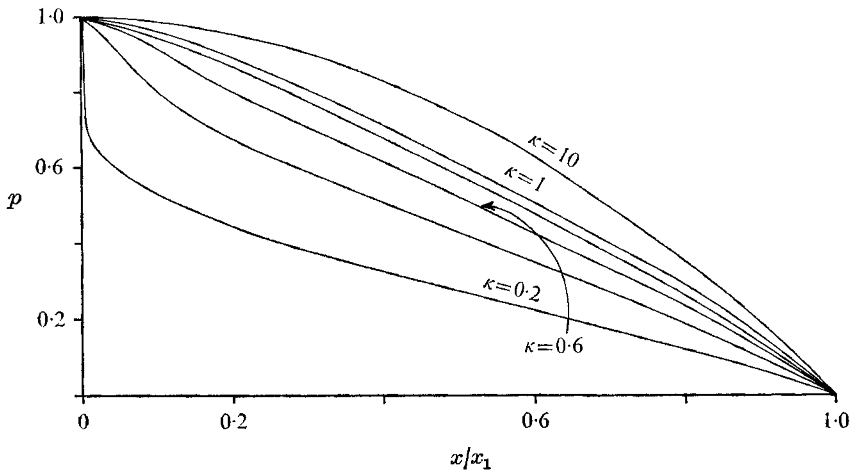

FIGURE 2. The variation of the pressure p in $0 \leqslant x \leqslant x_1$ for various values of the blowing parameter κ (see equation (3.9*b*)), where x_1 is given in table 2.

differential equation as follows. After integration of the right-hand side by parts and then differentiation of (4.7),

$$pG(p) = \frac{1}{\kappa\pi^{\frac{1}{2}}}(1-p)^{-\frac{1}{2}} - \frac{\kappa}{\pi^{\frac{1}{2}}}\int_p^1 \frac{G'(t)-(2\kappa\pi^{\frac{1}{2}})^{-1}(1-t)^{-\frac{3}{2}}}{(t-p)^{\frac{1}{2}}}\,dt,$$

which enables $G(q)$ to be substituted into the convolution integral in (4.7). We reverse the order of integration in the resulting double integral, put

$$q = (1-p)\sin^2\theta + t\cos^2\theta$$

and integrate with respect to θ, to find

$$p^{\frac{1}{2}}\int_p^1 qG(q)\,dq = 1-p^{\frac{1}{2}} - \kappa^2\int_p^1 \frac{G'(t)-(2\kappa\pi^{\frac{1}{2}})^{-1}(1-t)^{-\frac{3}{2}}}{t^{\frac{1}{2}}}\,dt.$$

Differentiating, multiplying by $p^{\frac{1}{2}}$ and again differentiating, we finally obtain

$$\kappa^2\frac{d^2G}{dp^2} + p^2\frac{dG}{dp} + \frac{5p}{2}G = \frac{3\kappa}{4(1-p)^{\frac{5}{2}}\pi^{\frac{1}{2}}}. \tag{4.11}$$

This second-order ordinary differential equation requires two boundary conditions for a unique solution and these are provided by (4.1) and (4.2), which give the initial behaviour of p as a function of x near $x = 0$, $p = 1$. A formal solution can be written down by the method of variation of parameters but generally (4.10) provides a more convenient method of finding x_1. The differential equation is particularly useful however when $\kappa \ll 1$. Then the forcing term of (4.11) is only relevant near $p = 1$; specifically, when $1-p \sim \kappa^2$ the function G has the asymptotic expansion

$$G = \pi\kappa^{-2}G_1 + G_2 + \dots \tag{4.12}$$

and (4.11), with initial conditions from (4.8) and (4.9), implies that

$$G_1 = \frac{2}{\pi} e^{u/\pi}\left(1 - \frac{1}{4}\int_\infty^u t^{-\frac{3}{2}} e^{-t/\pi}\,dt\right), \tag{4.13}$$

where $u = (1-p)\,\pi/\kappa^2$. The initial expansion (4.12) breaks down when $1-p = O(1)$ and to continue the solution for G to $p = 0$, the end point of the blow, only the complementary functions of (4.11) need be considered since G becomes exponentially large. Of these the significant one is that which decays exponentially as $p \to 1-$ for the match with (4.13) and it may be written as

$$G^{(1)}(p) = D \overset{**}{\int_1^\infty} e^{-p^3t/3\kappa^2} \frac{dt}{t^{\frac{1}{6}}(t-1)^{\frac{7}{6}}}. \tag{4.14}$$

Here D is a constant and ** means that only the finite part of the divergent integral is to be considered. When $p \approx 1$

$$G^{(1)}(p) \approx D(3\kappa^2)^{-\frac{1}{6}}(-\tfrac{7}{6})!\, e^{-p^3/3\kappa^2},$$

so that matching as $p \to 1 - O(\kappa^2)$ with (4.13) as $u \to \infty$ implies that

$$D = \frac{2(3\kappa^2)^{\frac{1}{6}}}{\kappa^2(-\frac{7}{6})!} e^{1/3\kappa^2}. \tag{4.15}$$

On integrating (4.14) we then find that

$$x_1 \approx \frac{4\pi^{\frac{1}{2}}}{3\kappa} e^{1/3\kappa^2} \quad \text{as} \quad \kappa \to 0. \tag{4.16}$$

A comparison between this formula and the numerical computations is also shown in table 2 and is seen to be favourable. In fact the comparison enables us to indicate a possible error in the numerical work at $\kappa = 0{\cdot}2$, where $2{\cdot}355$ should perhaps be replaced by $2{\cdot}358$, and to give a good estimate of the values of x_1 at $\kappa = 0{\cdot}1$.

From (4.16) we see that when $V_w^* Re^{\frac{3}{8}}/U_\infty^*$ is small

$$\delta_0 \approx \frac{3}{4Re^{\frac{1}{4}}} \lambda^{\frac{1}{2}} C^{\frac{1}{4}} (M_\infty^2 - 1)^{\frac{1}{4}} P_0(x_1^* - x_0^*)\, \kappa\pi^{\frac{1}{2}} e^{-1/3\kappa^2}, \tag{4.17}$$

where $\kappa = V_w \pi^{\frac{1}{2}}/(2P_0^3)^{\frac{1}{2}}$ and is defined in terms of V_w^* in (3.9b). Reference to table 2 shows that once V_w^* decreases below 4, x_1 increases very rapidly and hence so does $1/\delta_0^*$. On the other hand, we know from Smith & Stewartson (1973) that slot-injection implies separation in $x^* < x_0^*$ for large enough V_w and indeed it is likely that, if $(x_1^* - x_0^*)\,\epsilon^{-3}L^{-1}$ is large enough, separation occurs for *any* $V_w > 0$. It is instructive in the present context to inquire what happens to the approximations on which the present theory is based as $(x_1^* - x_0^*)/L \to 0$ and plate-injection approaches slot-injection. On converting to the triple-deck co-ordinates of § 2 we find that (3.12) reduces to

$$X_0 - X_s = (X_1 - X_0)/x_1, \tag{4.18}$$

where for the validity of the present theory $X_0 - X_s \gg \epsilon^{-1}$, from (3.3), while $X_1 - X_0$ is envisaged to be of $O(\epsilon^{-3})$ and must be $\gg \epsilon^{-1}$ if figure 1 (*b*) correctly describes the flow field near the onset of blowing. Thus the limits of validity of unseparated slot-blowing and fully separated plate-blowing do not overlap. For example if $V_w = 0{\cdot}8$, $\kappa \approx 0{\cdot}4$ and $x_1 \approx 45$, so that $(X_1 - X_0) = 45(X_0 - X_s)$. Hence if $X_1 - X_0$ is large and *finite* as in slot-blowing, $X_0 - X_s$ cannot be $\gg \epsilon^{-1}$ and so the present theory fails.

5. The gaps in the theory

It is convenient to consider the position that has now been reached in the theory of uniform injection into a supersonic laminar boundary layer across a part of a flat plate of finite length. If the injection begins at the leading edge and the velocity of injection is of $O(\epsilon^3 U_\infty^*)$ then the classical boundary-layer equations may be used, with merely a change in one of the boundary conditions at the wall, unless and until separation occurs, which must happen if the plate is long enough. The structure of the flow field near this kind of separation is at present an unsolved problem and indeed a doubt has been expressed (Smith & Stewartson 1973) as to whether the resulting adverse pressure gradient caused by the rapid thickening of the boundary layer in its neighbourhood might not cause the separation point to move upstream as far as the neighbourhood of the leading edge. With an increase of blowing rate to $O(\epsilon^3 U_\infty^*)$ separation must occur before the boundary-layer equations become appropriate and there is strictly no theory to account for the resulting flow. It is, however, very likely that some modified form of the Cole–Aroesty theory will apply over the major part of the plate.

If the injection begins at a finite distance x_0^* from the leading edge and the rate is of $O(\epsilon^4 U_\infty^*)$ it is anticipated that for a given rate the length of blowing needed to bring on separation is a decreasing function of x_0^*. The question (raised in Smith & Stewartson 1973) of whether separation, if it occurs, can only take place in the immediate neighbourhood of x_0^* as a free interaction remains an open one. When the rate of injection is of $O(\epsilon^3 U_\infty^*)$ the most satisfactory theory relates to slot-blowing and specifically to a blowing length of $O(\epsilon^3 L)$. In this case there is a pressure rise in a region of length also of $O(\epsilon^3 L)$ ahead of the blow, which then takes place in a largely favourable pressure gradient, unfavourable gradients occurring only in extreme conditions and being confined to the region near the onset of the blow. After the blow has ceased there is a final pressure rise which returns the pressure to its original value. This description of the pressure field in slot-injection is at variance with that which results from the classical theory of injection at rates of $O(\epsilon^4 U_\infty^*)$. There the most important adverse pressure gradients occur upstream of the blow and in the neighbourhood of separation and it has not yet proved possible to satisfy the terminal condition on the pressure. Consequently we are not satisfied with the classical theory.

The theory of slot-blowing is confined to blowing rates such that separation does not occur anywhere in the flow field. With increasing rates of injection it is probable that separation first occurs at x_0^* and thereafter a revised numerical procedure is needed which has yet to be devised. There is no mathematical objection to considering separation in this context. Increasing the slot length appears to have a slightly different effect in that separation first occurs just downstream of x_0^*.

The next class of problem studied is for injection rates of $O(\epsilon^3 U_\infty^*)$ over a finite part of the plate. If one accepts the implication of slot-injection, that the resulting flows separate by a free interaction ahead of the blow and that the separated flow has settled down to a fully developed state in which it is virtually at rest, it is possible to describe the flow field above the porous plate completely. The characteristic parameters of the flow field (blowing velocity and length of blow) are identical with those of slot-blowing and one can see in principle how the two studies can be joined but the necessary numerical work is quite formidable.

As the blowing rate increases to a rate of $O(U_\infty^*)$ the separation point moves upstream to the neighbourhood of the leading edge. The plateau pressure of the free interaction correspondingly increases and can in principle take on values independent of ϵ, which one might expect in view of the injection rates. The proposed flow field has something in common with that due to an aerodynamic spike but the details have not been worked out.

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Note added in proof. Contrary to the remark in Smith & Stewartson (1973, p. 20), Fernandez & Zukoski (1969) do report finding separation occurring upstream of the blow at higher blowing rates in turbulent conditions. Their observations in fact lend a measure of support to the theoretical model developed here.

Stewartson & Williams (1973) have been able to find a consistent asymptotic expansion of the solution of (2.2) on the lines suggested in §2.

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